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VP160 Final Big RC Part1

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Contents

- Oscillator
- Wave
- Gravitation and Celestial Motion

Harmonic Oscillator

Simple Harmonic Oscillator

$$\ddot{x} + \omega_0^2 x = 0 \qquad (1)$$

Damped Harmonic Oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \qquad (2)$$

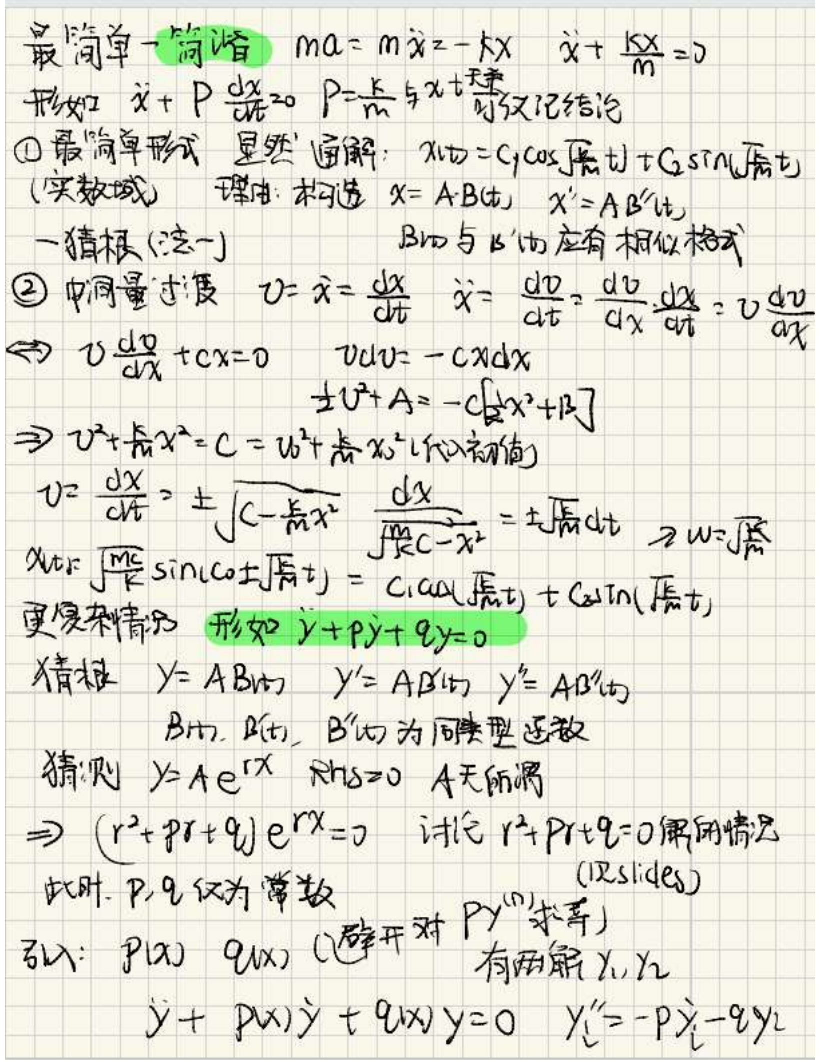
Driven Harmonic Oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \alpha\cos(\omega_d t) \qquad (3)$$

Definition: any system with the equation of motion of the below form is called a simple harmonic oscillator.

$$\ddot{x} + cx = 0$$

Solution about the Harmonic Oscillator (just copy it)



最简单—简谐 $ma = m\ddot{x} = -kx$ $\ddot{x} + \frac{kx}{m} = 0$

形如 $\ddot{x} + P\frac{dx}{dt} = 0$ $P = \frac{k}{m}$ 与 x 无关 可仅记结论

① 最简单形式 显然 通解: $x(t) = C_1\cos(\sqrt{\frac{k}{m}}t) + C_2\sin(\sqrt{\frac{k}{m}}t)$

(实数域) 理由: 构造 $x = A\cdot B(t)$ $x'' = AB''(t)$

—猜根 (法一) $B(t)$ 与 $B''(t)$ 应有 相似格式

② 中间量过渡 $v = \dot{x} = \frac{dx}{dt}$ $\ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$

$\Leftrightarrow v\frac{dv}{dx} + cx = 0$ $vdv = -cxdx$

$\frac{1}{2}v^2 + A = -c[\frac{1}{2}x^2 + B]$

$\Rightarrow v^2 + \frac{k}{m}x^2 = C = v_0^2 + \frac{k}{m}x_0^2$ (代入初值)

$v = \frac{dx}{dt} = \pm\sqrt{C - \frac{k}{m}x^2}$ $\frac{dx}{\sqrt{\frac{m}{k}C - x^2}} = \pm\sqrt{\frac{k}{m}}dt$ $\Rightarrow w = \sqrt{\frac{k}{m}}$

$x(t) = \sqrt{\frac{mC}{k}}\sin(C_0 \pm \sqrt{\frac{k}{m}}t) = C_1\cos(\sqrt{\frac{k}{m}}t) + C_2\sin(\sqrt{\frac{k}{m}}t)$

更复杂情况 形如 $\ddot{y} + p\dot{y} + qy = 0$

猜根 $y = AB(t)$ $y' = AB'(t)$ $y'' = AB''(t)$

$B(t)$, $B'(t)$, $B''(t)$ 为同类型函数

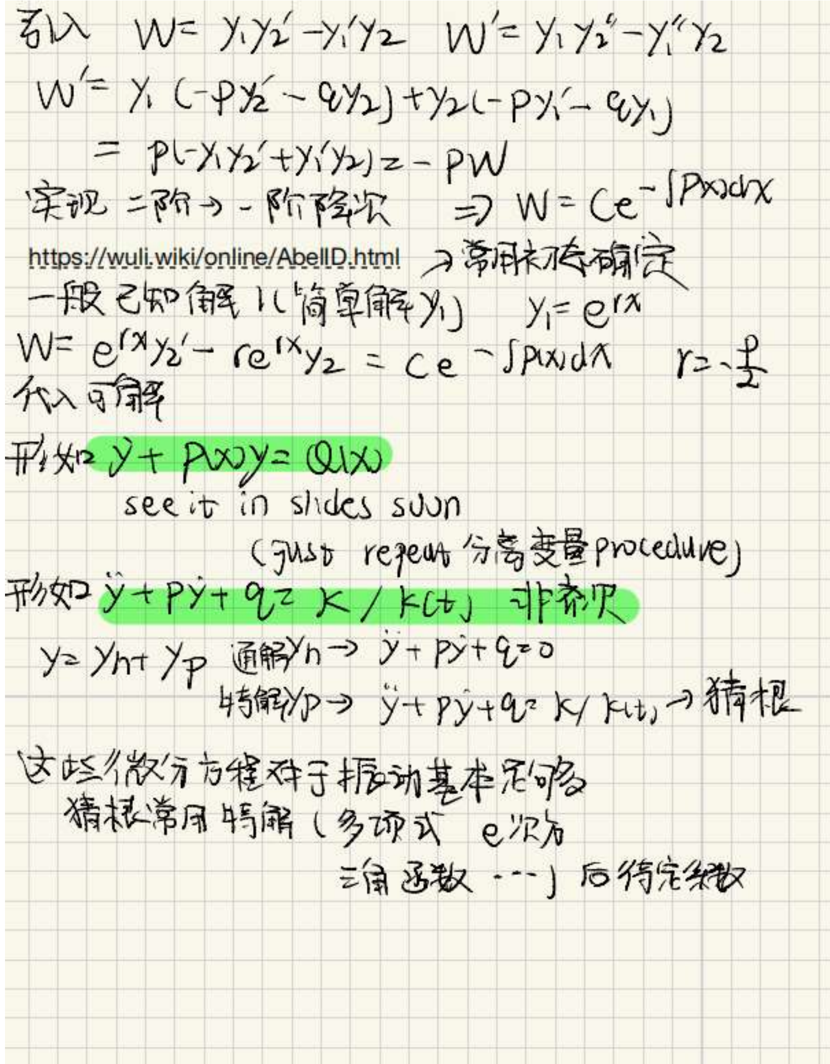
猜测 $y = Ae^{rx}$ $Rhs = 0$ A 无所谓

$\Rightarrow (r^2 + pr + q)e^{rx} = 0$ 讨论 $r^2 + pr + q = 0$ 解的情况 (见slides)

此时 p, q 仅为常数

引入: $p(x)$ $q(x)$ (避开对 $Py^{(n)}$ 求导) 有两解 y_1, y_2

$\ddot{y} + p(x)\dot{y} + q(x)y = 0$ $y_i'' = -py_i' - qy_i$



引入 $W = y_1y_2' - y_1'y_2$ $W' = y_1y_2'' - y_1''y_2$

$W' = y_1(-py_2' - qy_2) + y_2(-py_1' - qy_1)$

$= p(-y_1y_2' + y_1'y_2) = -pW$

实现 二阶 → 一阶降次 $\Rightarrow W = Ce^{-\int P(x)dx}$

https://wuli.wiki/online/AbelID.html → 常用初值确定

一般已知解1 (简单解 y_1) $y_1 = e^{rx}$

$W = e^{rx}y_2' - re^{rx}y_2 = Ce^{-\int p(x)dx}$ $r = -\frac{p}{2}$

代入可解

形如 $\dot{y} + P(x)y = Q(x)$

see it in slides soon

(just repeat 分离变量 procedure)

形如 $\ddot{y} + p\dot{y} + q = k\ /\ k(t)$ 非齐次

$y = y_h + y_p$ 通解 $y_h \rightarrow \ddot{y} + p\dot{y} + q = 0$

特解 $y_p \rightarrow \ddot{y} + p\dot{y} + q = k\ /\ k(t) \rightarrow$ 猜根

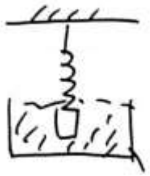
这些微分方程对于振动基本足够

猜根常用特解 (多项式 e次方 三角函数 ……) 后待定系数

From "Solutions to differential equations in vibration (summary).pdf" in canvas

Continue: (just copy)

Damped Oscillations.



$$m\ddot{x} = -b\dot{x} - kx$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \omega_0^2 x = 0 \quad \text{where } \omega_0^2 = \frac{k}{m}.$$

we now know how to solve it!

① $p^2 > 4q \iff \left(\frac{b}{m}\right)^2 > 4\omega_0^2$

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \iff x(t) = C_1 e^{-\left(\frac{b}{2m} + \sqrt{\frac{b^2}{4m^2} - \omega_0^2}\right)t} + C_2 e^{-\left(\frac{b}{2m} - \sqrt{\frac{b^2}{4m^2} - \omega_0^2}\right)t}$$

in this case, no periodic behavior, x(t) decays w.r.t time. exponentially

this is called overdamped regime

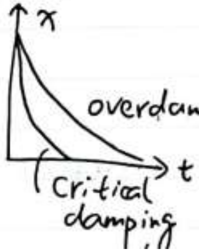
② $p^2 < 4q \iff \left(\frac{b}{m}\right)^2 < 4\omega_0^2$

$$y = C_1 e^{\lambda x}\cos\mu x + C_2 e^{\lambda x}\sin\mu x \iff x(t) = Ae^{-\frac{b}{2m}t}\cos\left(\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}\,t + \varphi\right)$$

③ critical damping

$$x(t) = D_1 e^{-\frac{b}{2m}t} + D_2 t e^{-\frac{b}{2m}t}$$

No periodic behavior. The system aperiodically returns to the equilibrium position.



Critically damping refers to the damping level where the system achieves the fastest return to equilibrium without oscillation.

Wave:

Classical Wave Equation

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{\partial^2 \xi}{v_p^2 \partial t^2} = 0 \qquad (6)$$

Phase Velocity

$$v_p = \frac{\lambda}{T} = \frac{\omega}{k} = \frac{\omega\lambda}{2\pi} = \sqrt{\frac{T}{\rho}} \text{ (on a string)} \qquad (7)$$

Complex Wave Function

$$\tilde{\xi} = \tilde{A} e^{i(kx - \omega t)} \qquad (8)$$

Rate of Energy Transmission

$$P_{avg} = \frac{1}{2} \rho v \omega^2 A^2 \qquad (9)$$

Wave: the disturbance of a medium propagating through space

Transverse wave: the direction of displacement of medium particles is perpendicular to the direction of wave propagation.

Longitudinal wave: the direction of displacement of medium particles is parellel with the direction of wave propagation.

Basic Concept:

Sinusoidal (Harmonic) Waves:
A propagating mechanical wave in the shape of a cosine or sine form is called a harmonic wave. The disturbance of the medium is :

$$\xi = \xi 0 \; \mathrm{Cos}\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right)$$

$$K = \frac{2\pi}{\lambda} \text{ wave number}$$

$$\omega = \frac{2\pi}{T} \; frequency$$

Normal Mode:

Coupled Oscillators

$$\begin{cases} \ddot{x_1} + a_1\dot{x_1} + b_1\dot{x_2} + c_1x_1 + d_1x_2 = 0 \\ \ddot{x_2} + a_2\dot{x_1} + b_2\dot{x_2} + c_2x_1 + d_2x_2 = 0 \end{cases} \quad (10)$$

Standing Waves

$$\xi(x,t) = 2A\sin\left(\frac{2\pi x}{\lambda}\right)\cos(\omega t) \quad (11)$$

A standing wave is formed by the interference of two waves of the same frequency and amplitude traveling in opposite directions.

The characteristic feature of a standing wave is the presence of fixed nodes and antinodes, where nodes are points of zero amplitude and antinodes are points of maximum amplitude.

Gravitation

Source: any mass
The gravitation is much weaker than the other three Fundamental interactions

$$F_{12} = -G\frac{m_1 m_2}{r_{12}2}\frac{r_{12}}{|r_{12}|}$$

$$G = 6.67 \times 10^{-11}[N \cdot \frac{m^2}{kg}]$$

Gravitational Field

$$E = \frac{F}{m} = -G\frac{M}{r^2}\frac{r}{|r|}$$

Gravitational Potential Energy

We always set the infinite far place as zero potential. Then the gravitational potential energy equals the work done by gravitation from infinite far place to now

$$U_p = -\frac{GMm}{r}$$

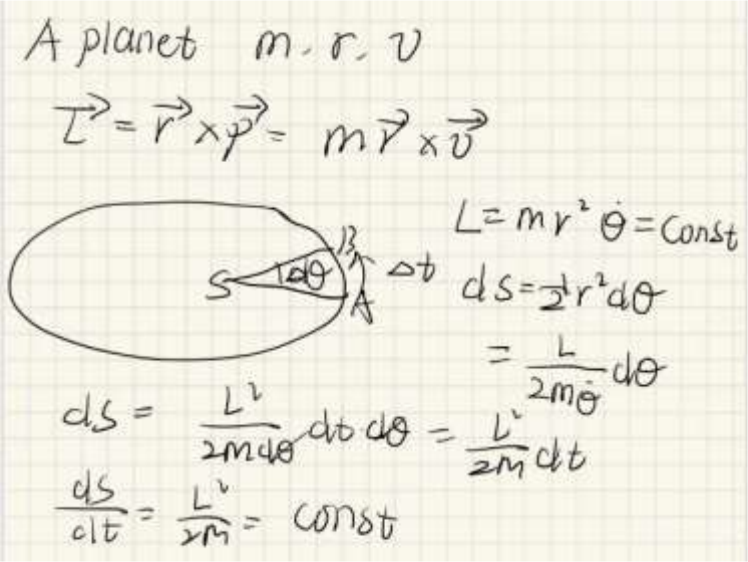
Then we can have first cosmic velocity, second cosmic velocity and the third cosmic velocity

Kapler’s Law 1 and 2

1: Each planet moves in an elliptical orbit with the Sun at on of the focal point.

2:The line from the Sun to the given planet sweeps out equals areas in equal time.

If at time Δt, The area swept by the line between the planet and the sun in Δt is ΔA. Then this swept area is constant over any equal interval of time.



$$\frac{\Delta A}{\Delta t} \text{ is constant}$$

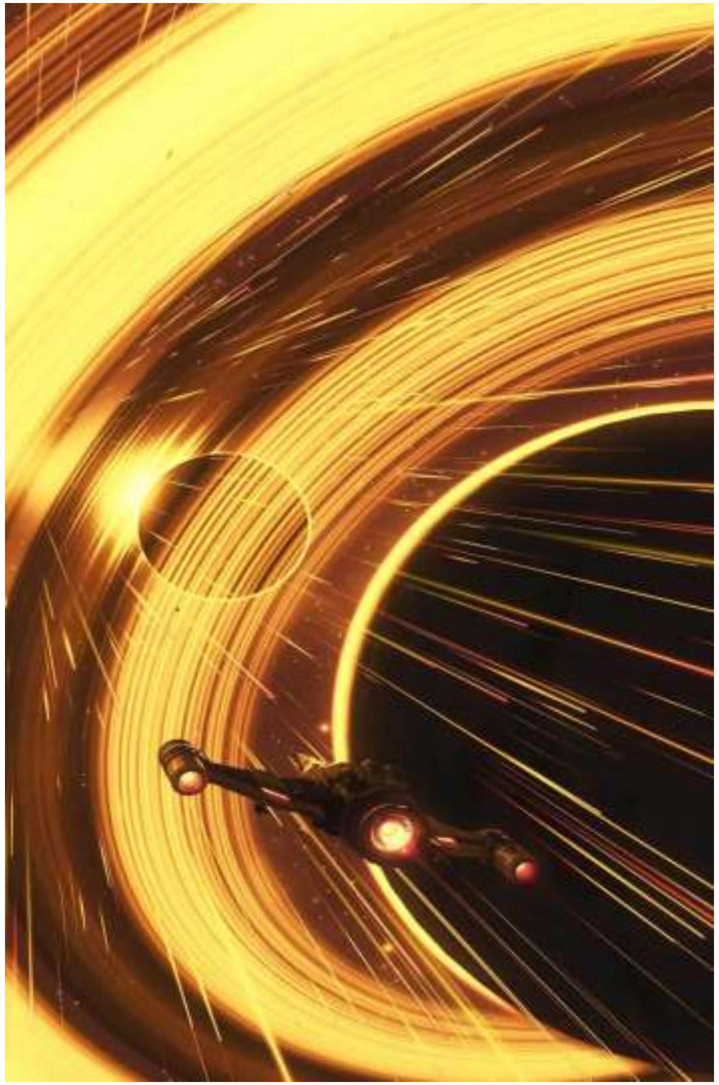
Kapler’s Law 3

3:The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM}}$$

$$\frac{R_1}{R_2} = \left(\frac{T_1}{T_2}\right)^{\frac{2}{3}}, \frac{T_1}{T_2} = \left(\frac{R_1}{R_2}\right)^{\frac{3}{2}}$$

Where, m1m2 is the mass of the two corresponding planets, T1T2 is the period of the two corresponding planets moving around the same star, and R1R2 is the average orbital radius of the two planets moving around the same star



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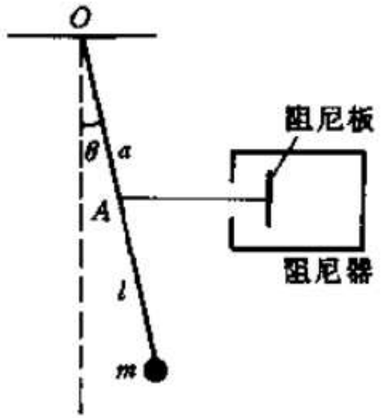
Simple Questions:

Problem 29: As shown in the figure, a thin rod of length l is suspended from its upper end at point O, with a mass m attached to the lower end. The mass can swing without friction around the axis O. Now, a damper is connected to point A on the rod via a connecting rod. The damping force f acts horizontally, and its magnitude is proportional to the speed v of the damper plate, i.e., $f = \gamma v$, where γ is the damping coefficient. Let $OA = a$. The masses of the rod, connecting rod, and damper plate are negligible compared to m. Assume weak damping. Calculate the logarithmic decrement of the small amplitude damped oscillation and the percentage of energy lost per period.



力图 7－29－1

【题 29】 如图，细杆长 l，上端 O 悬挂，下端与质量为 m 的摆球相连，摆球可绕 O 轴无摩擦地摆动．今在细杆的 A 点经连杆接一阻尼器，阻尼力 f 沿水平方向，其大小与阻尼板的速度 v 成正比，即 $f=\gamma v$，γ 为阻力系数．设 $OA=a$，细杆、连杆、阻尼板等的质量均远小于 m，可忽略不计，阻尼较弱．试求小幅阻尼振动的对数减缩，以及一周期内损失能量的百分比．

O $\sqrt{2}\,\omega_0$ $\sqrt{2}\,\omega_0$ ω_0

力图 7－28－8

Solution :

【解】 对于小幅摆动，$\cos\theta\approx1$，$\sin\theta\approx0$，由转动定理

$$I\ddot{\theta}=-mgl\theta-\gamma a v_A$$

式中

$$I=ml^2,\quad v_A=a\dot{\theta}$$

代入，得

$$ml^2\ddot{\theta}=-mgl\theta-\gamma a^2\dot{\theta}$$

即

$$\ddot{\theta}+\frac{\gamma a^2}{ml^2}\dot{\theta}+\frac{g}{l}\theta=0$$

令

$$2\beta=\frac{\gamma a^2}{ml^2},\quad \omega_0^2=\frac{g}{l}$$

则

$$\ddot{\theta}+2\beta\dot{\theta}+\omega_0^2\theta=0$$

这是阻尼振动的运动方程，它的解为

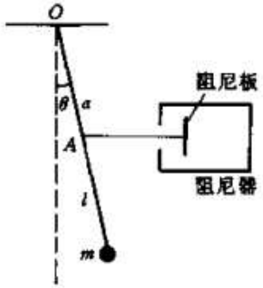
$$\theta=A\,e^{-\beta t}\cos(\omega t+\varphi) \qquad (1)$$

式中 β 为阻尼系数，ω 是振动圆频率，为

$$\omega=\sqrt{\omega_0^2-\beta^2}$$

(1)式表明，在 t 时刻和 $(t+T)$ 时刻的振幅分别为 $Ae^{-\beta t}$ 和 $Ae^{-\beta(t+T)}$，其中 $T=\frac{2\pi}{\omega}$ 是周期．因此，阻尼振动的对数减缩为

$$\delta=\ln\left[\frac{e^{-\beta t}}{e^{-\beta(t+T)}}\right]=\ln e^{\beta T}=\beta T=\beta\frac{2\pi}{\omega}$$



力图 7－29－1

$$=\frac{2\pi\beta}{\sqrt{\omega_0^2-\beta^2}}=\frac{2\pi}{\sqrt{\left(\frac{\omega_0}{\beta}\right)^2-1}}=\frac{2\pi\gamma a^2}{\sqrt{4m^2l^3g-\gamma^2a^4}}$$

由(1)式，

$$\dot{\theta}=-A\,e^{-\beta t}[\beta\cos(\omega t+\varphi)+\omega\sin(\omega t+\varphi)] \qquad (2)$$

任意时刻 t，振动系统的机械能 E 为

$$E=\frac{1}{2}m(l\dot{\theta})^2+mgl(1-\cos\theta)=\frac{1}{2}ml^2\dot{\theta}^2+\frac{1}{2}mgl\theta^2$$

把(1)、(2)式代入，并注意到 $\omega_0^2=\frac{g}{l}$ 和 $\omega_0^2=\omega^2+\beta^2$，得

$$E=\frac{1}{2}ml^2A^2e^{-2\beta t}[\omega^2+2\beta^2\cos^2(\omega t+\varphi)+\beta\omega\sin2(\omega t+\varphi)]$$

在 β 很小时可简化为

$$E\approx\frac{1}{2}ml^2A^2\omega^2e^{-2\beta t}$$

一周期 T 内的能量损失为

$$|\Delta E|=\frac{1}{2}ml^2A^2\omega^2[e^{-2\beta t}-e^{-2\beta(t+T)}]$$

$$=\frac{1}{2}ml^2A^2\omega^2e^{-2\beta t}(1-e^{-2\beta T})\approx\frac{1}{2}ml^2A^2\omega^2e^{-2\beta T}\cdot2\beta T$$

故能量损失百分比为

$$\frac{|\Delta E|}{E}=2\beta T=2\delta=\frac{4\pi\gamma a^2}{\sqrt{4m^2l^3g-\gamma^2a^4}}$$

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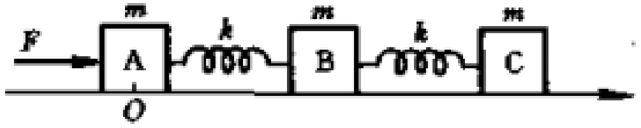
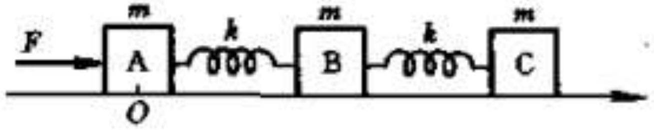
Challenge Problems:

As shown in the figure, three blocks, each with mass m, are placed on a frictionless horizontal surface. They are connected by two springs with spring constant k and natural length x_0. Initially, the three blocks are stationary, with block A located at the origin O. A periodic force F is applied to block A with the form $F = F_0 \cos \omega t$, acting in the horizontal direction and along the line connecting the three blocks. Determine the characteristic frequencies of the vibration system and derive the equation of motion for block C.



【题 32】 如图，质量均为 m 的三个滑块放置在光滑水平桌面上，彼此以原长为 x_0、劲度系数为 k 的二个相同弹簧连接起来．开始时三滑块静止，滑块 A 位于原点 O，两弹簧均为原长．对滑块 A 施以周期性外力 F，已知 $F = F_0 \cos\omega t$，在水平方向并沿三滑块连线．试求振动系统的特征频率，并写出滑块 C 的运动方程．



力图 7－32－1

Long Long Solution:

【分析】 根据牛顿第二定律分别写出三滑块的动力学方程．引进简正坐标后，可得出三个独立的受迫振动方程，于是特征频率可得．解方程，并回到原来的坐标，可得滑块 C 的运动方程．

【解】 设在任意时刻 t，A、B、C 的坐标分别为 x_A、x_B、x_C，由牛顿第二定律，三滑块遵守的动力学方程为

$$\text{滑块 A}\quad m\ddot{x}_A = F_0\cos\omega t + k(x_B - x_A - x_0) \tag{1}$$

$$\text{滑块 B}\quad m\ddot{x}_B = k(x_C - x_B - x_0) - k(x_B - x_A - x_0) \tag{2}$$

$$\text{滑块 C}\quad m\ddot{x}_C = -k(x_C - x_B - x_0) \tag{3}$$

三式相加，得

$$m(\ddot{x}_A + \ddot{x}_B + \ddot{x}_C) = F_0\cos\omega t \tag{4}$$

(1)式减(3)式，得

$$m(\ddot{x}_A - \ddot{x}_C) + k(x_A - x_C) = F_0\cos\omega t - 2kx_0 \tag{5}$$

(1)式减 2 乘(2)式加(3)式，得

$$m(\ddot{x}_A - 2\ddot{x}_B + \ddot{x}_C) + 3k(x_A - 2x_B + x_C) = F_0\cos\omega t \tag{6}$$

引进如下简正坐标

$$\begin{cases} q_1 = x_A + x_B + x_C \\ q_2 = x_A - x_C \\ q_3 = x_A - 2x_B + x_C \end{cases} \tag{7}$$

方程(4)、(5)、(6)式变为

$$m\ddot{q}_1 = F_0\cos\omega t$$

$$m\ddot{q}_2 + kq_2 = F_0\cos\omega t - 2kx_0$$

$$m\ddot{q}_3 + 3kq_3 = F_0\cos\omega t$$

$$\begin{cases} \ddot{q}_1 = \dfrac{F_0}{m}\cos\omega t \\ \ddot{q}_2 + \dfrac{k}{m}q_2 = \dfrac{F_0}{m}\cos\omega t - 2\dfrac{k}{m}x_0 \\ \ddot{q}_3 + 3\dfrac{k}{m}q_3 = \dfrac{F_0}{m}\cos\omega t \end{cases} \tag{8}$$

从上述可知，系统的振动频率除周期性力的圆频率 ω 外，还有下述系统的特征圆频率

$$\omega_2 = \sqrt{\frac{k}{m}},\quad \omega_3 = \sqrt{\frac{3k}{m}}$$

方程(8)式可改写为

$$\ddot{q}_1 = \frac{F_0}{m}\cos\omega t \tag{9}$$

$$\ddot{q}_2 + \omega_2^2 q_2 = \frac{F_0}{m}\cos\omega t - 2\omega_2^2 x_0 \tag{10}$$

$$\ddot{q}_3 + \omega_3^2 q_3 = \frac{F_0}{m}\cos\omega t \tag{11}$$

(i)解方程(9)式，(9)式经两次积分后，得

$$q_1 = -\frac{F_0}{m\omega^2}\cos\omega t + C_1 t + C_2$$

C_1 和 C_2 为积分常量，由初条件决定．初条件为

$$t=0\text{ 时},\ x_A = 0,\ x_B = x_C,\ x_C = 2x_0$$

$$\dot{x}_A = \dot{x}_B = \dot{x}_C = 0$$

利用定义(7)式，关于简正坐标的初条件为

$$t=0\text{ 时},\begin{cases} q_1 = 3x_0,\ q_2 = -2x_0,\ q_3 = 0 \\ \dot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0 \end{cases} \tag{12}$$

由上述初条件，得出

$$C_1 = 0$$

$$C_2 = 3x_0 + \frac{F_0}{m\omega^2}$$

于是，得

$$q_1 = \frac{F_0}{m\omega^2}(1 - \cos\omega t) + 3x_0 \tag{13}$$

(ii)解方程(10)式，(10)式可改写为

$$\ddot{q}_2 + \omega_2^2(q_2 + 2x_0) = \frac{F_0}{m}\cos\omega t$$

引入

$$Q = q_2 + 2x_0$$

上式简化为

Continue:

$$\ddot{Q}+\omega_2^2 Q=\frac{F_0}{m}\cos\omega t \qquad (14)$$

这是无阻尼的受迫振动方程，设其特解为

$$Q_1=A\cos\omega t$$

代入方程(14)式，比较系数，得

$$A=\frac{F_0}{m(\omega_2^2-\omega^2)}$$

于是特解为

$$Q_1=\frac{F_0}{m(\omega_2^2-\omega^2)}\cos\omega t$$

方程(14)式对应的齐次方程的通解为

$$Q_2=C_1\cos\omega_2 t+C_2\sin\omega_2 t$$

故方程(14)式的通解为

$$Q=\frac{F_0}{m(\omega_2^2-\omega^2)}\cos\omega t+C_1\cos\omega_2 t+C_2\sin\omega_2 t$$

于是得

$$q_2=Q-2x_0=\frac{F_0}{m(\omega_2^2-\omega^2)}\cos\omega t+C_1\cos\omega_2 t+C_2\sin\omega_2 t-2x_0$$

积分常量 C_1 和 C_2 由初条件(12)式确定，为

$$C_1=-\frac{F_0}{m(\omega_2^2-\omega^2)},\quad C_2=\frac{2x_0}{\omega_2}$$

最后得

$$q_2=\frac{F_0}{m(\omega_2^2-\omega^2)}(\cos\omega t-\cos\omega_2 t)+\frac{2x_0}{\omega_2}\sin\omega_2 t-2x_0 \qquad (15)$$

(iii)解方程(11)式，同理，通解为

$$q_3=\frac{F_0}{m(\omega_3^2-\omega^2)}\cos\omega t+C_1'\cos\omega_3 t+C_2'\sin\omega_3 t$$

由初条件(12)式，得

$$C_1'=-\frac{F_0}{m(\omega_3^2-\omega^2)},\quad C_2'=0$$

故

$$q_3=\frac{F_0}{m(\omega_3^2-\omega^2)}(\cos\omega t-\cos\omega_3 t) \qquad (16)$$

由(7)式，可解得

$$x_C=\frac{q_1}{3}-\frac{q_2}{2}+\frac{q_3}{6}$$

把(13)、(15)、(16)式代入，最后得滑块 C 的运动方程为

$$x_C=2x_0+\frac{F_0}{3m\omega^2}(1-\cos\omega t)-\frac{F_0}{2m(\omega_2^2-\omega^2)}(\cos\omega t-\cos\omega_2 t)-\frac{x_0}{\omega_2}\sin\omega_2 t+\frac{F_0}{6m(\omega_3^2-\omega^2)}(\cos\omega t-\cos\omega_3 t)$$

式中 ω_2 和 ω_3 是系统的特征频率。

Tips:

When you encounter such problems with more than 3 objects, what you need is just your patience.

And to simplify your calculation, you can use new variable from the relationship between original variables.

Like this:

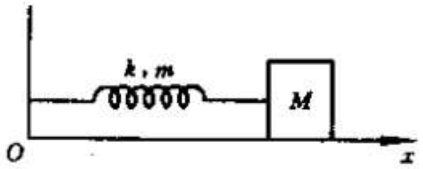
$$\begin{cases} q_1=x_A+x_B+x_C \\ q_2=x_A-x_C \\ q_3=x_A-2x_B+x_C \end{cases}$$

Wave Comprehensive Questions:

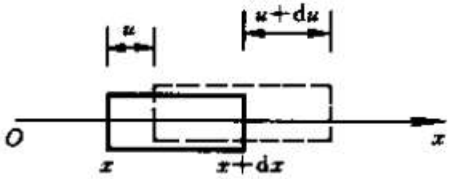
Problem 36 As shown in Figure 7-36-1, a spring with a mass m, spring constant k, and natural length L has one end fixed while the other end is connected to a mass M. The entire system is placed horizontally, and the mass M can slide on a frictionless horizontal surface.

1. Derive the wave equation for the longitudinal elastic wave and find the wave speed.
2. Assuming L is much smaller than the wavelength of the elastic wave and the amplitude of M is A_0, find the amplitude distribution along the spring.
3. Under the condition $m \ll M$, determine the zeroth-order and first-order approximate frequencies.

【题 36】 如力图 7－36－1，质量为 m、劲度系数为 k 的弹簧原长为 L，一端固定，另一端与一质量为 M 的物体相连，整个系统水平放置，物体 M 可在光滑水平面上滑动 .1. 试导出纵向弹性波的波动方程并求波速 .2. 假定 L 远小于弹性波的波长，M 的振幅为 A_0，试求弹簧上各点的振幅分布 .3. 在 $m \ll M$ 的条件下，试分别求零级近似和一级近似的频率 .



力图 7－36－1



力图 7－36－2

Solution:

【分析】 弹簧质量不可忽略时,弹簧上不同地点的弹性力将有一定分布,首先求出这一分布．如力图 7-36-2,以弹簧固定端为坐标原点 O,水平方向为 x 轴．考虑弹簧上介于 x 和 $(x+\mathrm{d}x)$ 之间的一小段．弹簧运动时,设 x 点的位移为 u,$(x+\mathrm{d}x)$ 点的位移为 $(u+\mathrm{d}u)$,则该 $\mathrm{d}x$ 小段的伸长形变为 $\mathrm{d}u$,相对形变为 $\frac{\mathrm{d}u}{\mathrm{d}x}$．由胡克定律,该 $\mathrm{d}x$ 小段两端的弹性力为

$$F = Y\frac{\mathrm{d}u}{\mathrm{d}x} \tag{1}$$

式中 Y 是弹簧的弹性模量．Y 与劲度系数 k 密切相关．设整个弹簧伸长 Δx,则弹性力

$$F = k\Delta x = Y\frac{\Delta x}{L}$$

故

$$Y = kL$$

弹簧运动时,不同地点的相对形变 $\frac{\mathrm{d}u}{\mathrm{d}x}$ 各不相同,由(1)式,不同地点的弹性力也各异．

考虑弹簧上一小段,利用(1)式,根据牛顿第二定律即可得到波动方程,波动方程中包含了波速 v,因而波速可求．

弹簧上各部分均作同频率的简谐振动,写出所得波动方程的猜测解,代入波动方程,再利用边界条件,即可求得弹簧上的振幅分布．

频率 ω 的零级近似应等于忽略弹簧质量时的值, 级近似应等于本章题 7 的结果．

【解】 1. 如力图 7-36-3,考虑弹簧上介于 x 和 $(x+\Delta x)$ 之间的一小段,它的左、右两端分别受弹性力 F_1 和 F_2,利用(1)式,该小段所受合力为

$$F_2 - F_1 = Y\left[\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x+\Delta x} - \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_x\right]$$

展成级数,忽略高级小量,得

$$F_2 - F_1 = Y\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\cdot\Delta x = Y\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\Delta x$$

该小段的质量为 $\frac{m}{L}\Delta x$,由牛顿第二定律

$$Y\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\Delta x = \frac{m}{L}\Delta x\frac{\mathrm{d}^2u}{\mathrm{d}t^2}$$

力图 7-36-3

即

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{m}{YL}\frac{\mathrm{d}^2u}{\mathrm{d}t^2}$$

写成偏导数形式,

$$\frac{\partial^2u}{\partial x^2} - \frac{m}{YL}\frac{\partial^2u}{\partial t^2} = 0$$

或

$$\frac{\partial^2u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2u}{\partial t^2} = 0 \tag{2}$$

这是波动方程的标准形式,式中 v 即波速,为

$$v = \sqrt{\frac{YL}{m}} = \sqrt{\frac{k}{m}}L \tag{3}$$

Maybe the most important part

Continue:

2. 简谐波是波动方程(2)式的解．弹簧上各处均以相同频率 ω 振动，因弹簧全长 L 比波长小得多，故各处振动的相位可看作相同，不同地点的振幅各不相同．根据以上考虑，可写出方程(2)的猜测解，为

$$u(x,t)=A(x)\cos(\omega t+\varphi) \tag{4}$$

代入波动方程(2)，注意初相位 φ 已看作恒量，得

$$\frac{\mathrm{d}^2A(x)}{\mathrm{d}x^2}+\frac{\omega^2}{v^2}A(x)=0$$

上式具有简谐振动微分方程的形式，其通解为

$$A(x)=a\sin\left(\frac{\omega}{v}x\right)+b\cos\left(\frac{\omega}{v}x\right)$$

式中 a 和 b 为待定系数，可由边界条件确定．在弹簧左端，$x=0$，$A(0)=0$；在弹簧右端，$x=L$，$A(L)=A_0$，故有

$$0=b\cos\left(\frac{\omega}{v}L\right)$$

$$A_0=a\sin\left(\frac{\omega}{v}L\right)+b\cos\left(\frac{\omega}{v}L\right)$$

由以上两式，得出

$$b=0$$

$$a=\frac{A_0}{\sin\left(\frac{\omega}{v}L\right)}$$

故得振幅分布为

$$A(x)=\frac{A_0}{\sin\left(\frac{\omega}{v}L\right)}\sin\left(\frac{\omega}{v}x\right)$$

这是一个正弦函数，式中的波速 v 由(3)式确定，圆频率 ω 将在下面求出．将 $A(x)$ 代入(4)式，有

$$u(x,t)=a\sin\left(\frac{\omega}{v}x\right)\cos(\omega t+\varphi) \tag{5}$$

Continue:

3. 考虑物体 M，由(1)式，所受弹性力为 $Y\left(\frac{\partial u}{\partial x}\right)_L$，脚标 L 表示 $x=L$ 处的弹性力，加速度为 $\left(\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}\right)_L$，由牛顿第二定律，

$$M\left(\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}\right)_L = -Y\left(\frac{\partial u}{\partial x}\right)_L \tag{6}$$

由(5)式

$$\left(\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}\right)_L = -a\omega^2\sin\left(\frac{\omega}{v}L\right)\cos(\omega t+\varphi)$$

$$\left(\frac{\partial u}{\partial x}\right)_L = \frac{a\omega}{v}\cos\left(\frac{\omega}{v}L\right)\cos(\omega t+\varphi)$$

代入(6)式，得

$$-Ma\omega^2\sin\left(\frac{\omega}{v}L\right) = -Y\frac{a\omega}{v}\cos\left(\frac{\omega}{v}L\right)$$

即

$$M\omega\tan\left(\frac{\omega}{v}L\right) = \frac{Y}{v}$$

或

$$\frac{\omega L}{v}\tan\left(\frac{\omega}{v}L\right) = \frac{YL}{Mv^2}$$

因

$$Y = kL,\ v^2 = \frac{k}{m}L^2$$

故得

$$\frac{\omega L}{v}\tan\left(\frac{\omega}{v}L\right) = \frac{m}{M} \tag{7}$$

系统的振动圆频率 ω 由(7)式决定．因 $m \ll M$，$\frac{\omega L}{v}$ 是很小的量，取零级近似，有

$$\tan\left(\frac{\omega L}{v}\right) \approx \frac{\omega L}{v}$$

故

$$\frac{\omega^2 L^2}{v^2} = \frac{m}{M}$$

$$\omega^2 = \frac{m}{M}\frac{v^2}{L^2} = \frac{m}{M}\frac{k}{m} = \frac{k}{M}$$

即

$$\omega = \sqrt{\frac{k}{M}}$$

这正是忽略弹簧质量时的振动圆频率．

取一级近似，有

$$\tan\left(\frac{\omega L}{v}\right) = \frac{\omega L}{v} + \frac{1}{3}\left(\frac{\omega L}{v}\right)^3$$

代入(7)式，

$$\frac{\omega L}{v}\left[\frac{\omega L}{v} - \frac{1}{3}\left(\frac{\omega L}{v}\right)^3\right] = \frac{m}{M}$$

即

$$\left(\frac{\omega L}{v}\right)^2\left[1+\frac{1}{3}\left(\frac{\omega L}{v}\right)^2\right] = \frac{m}{M}$$

或

$$\omega^2 = \left(\frac{v}{L}\right)^2\frac{m}{M}\frac{1}{1+\frac{1}{3}\omega^2\left(\frac{L}{v}\right)^2}$$

因 $\left(\frac{v}{L}\right)^2 = \frac{k}{m}$，上式右端的 ω 用 $\omega^2 = \frac{k}{M}$ 近似，得

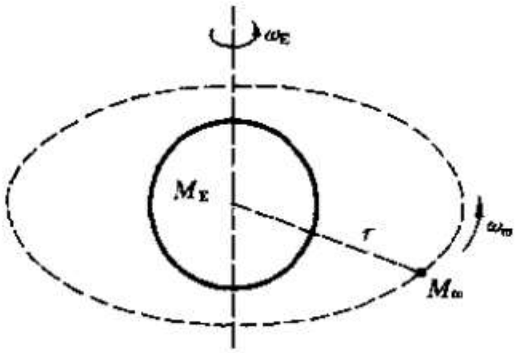
$$\omega^2 = \frac{k}{M}\frac{1}{1+\frac{1}{3}\frac{k}{M}\frac{m}{k}} = \frac{k}{M}\frac{1}{1+\frac{1}{3}\frac{m}{M}} = \frac{k}{M+\frac{m}{3}}$$

$$\omega = \sqrt{\frac{k}{M+\frac{m}{3}}}$$

此即本章题 7 中的结果．

Celestial Motion Question:

【题 7】 如图,地球和月球的质量分别为 M_E 和 M_m,两者中心的距离用 r 表示,月球可看作质点.设地球的自转轴和月球绕地球作圆周运动的转轴重合,都通过地球中心,地球和月球的角速度分别为 ω_E 和 ω_m.由于潮汐运动,地球自转角速度会发生微小变化,从而地球和月球的间距以及地球和月球系统的总能量也将发生变化.



力图 6-7-1

1. 试证明,地球和月球间距的时间变化率 $\dot{r}$ 与地球自转角速度的时间变化率 $\dot{\omega}_E$ 的关系为

$$\dot{r} = -\frac{2\sqrt{r}\,I_E}{M_m\sqrt{GM_E}}\dot{\omega}_E$$

式中 I_E 是地球对自转轴的转动惯量,G 是引力常量.

2. 试导出地球和月球系统机械能时间变化率的表达式.

3. 设现在的地球与月球间距为 $r(0)$,地球的自转角速度为 $\omega_E(0)$,受扰动后相应的量变为 $r(t)$ 和 $\omega_E(t)$,因扰动很小,月球仍作圆周运动.试证明

$$\omega_E(t) = \omega_E(0) - \frac{M_m\sqrt{GM_E}}{I_E}\left[\sqrt{r(t)} - \sqrt{r(0)}\right]$$

已知 $G = 6.672\times10^{-11}\ \mathrm{N\cdot m^2/kg^2}$, $M_E = 5.89\times10^{24}$ kg, $M_m = 7.34\times10^{22}$ kg, $r = 3.84\times10^{8}$ m, $R_E = 6.37\times10^{6}$ m(地球半径).试计算地球和月球间距增加 2% 时,地球日将变成多少小时.

In the figure, the masses of the Earth and the Moon are denoted by M_E and M_m, respectively, and the distance between their centers is represented by r. The Moon can be considered a point mass. The rotational axes of both the Earth and the Moon pass through the Earth's center. The angular velocities of the Earth and the Moon are ω_E and ω_m, respectively. Due to tidal forces, the Earth's rotation speed undergoes a slight change, which in turn affects the distance between the Earth and the Moon, as well as the total energy of the Earth-Moon system.

1. Prove the relationship between the rate of change of the distance between the Earth and the Moon, $\dot{r}$, and the rate of change of the Earth's angular velocity, $\dot{\omega}_E$:
   $$\dot{r} = -\frac{2\sqrt{r}I_E}{M_m\sqrt{GM_E}}\dot{\omega}_E$$
   where I_E is the Earth's moment of inertia with respect to its own rotation, and G is the gravitational constant.
2. Derive the expression for the rate of change of the total angular momentum of the Earth-Moon system.
3. Assuming the distance between the Earth and the Moon is $r(0)$ and the initial angular velocity of the Earth's rotation is $\omega_E(0)$, show that after being perturbed, the resulting expressions for $r(t)$ and $\omega_E(t)$ are:
   $$\omega_E(t) = \omega_E(0) - \frac{M_m}{M_E}\sqrt{\frac{GM_E}{r(0)}}\left(\sqrt{\frac{r(t)}{r(0)}} - \sqrt{\frac{r(0)}{r(t)}}\right)$$

   Given:
   $G = 6.672\times10^{-11}\mathrm{N\cdot m^2/kg^2}, M_E = 5.89\times10^{24}\mathrm{kg}, M_m = 7.34\times10^{22}\mathrm{kg}, r = 3.84\times10^{8}\mathrm{m}, N_r = 6.37\times10^{6}\mathrm{m}$

   If the distance between the Earth and the Moon increases by 2%, how many hour ↓ ill the Earth's rotation period

Solution:

【分析】 地、月系统不受外力矩，系统的角动量守恒，它给出了 ω_E、ω_m、r 的关系．月球绕地球作圆周运动的向心力为地球引力，所以 ω_m 与 r 有关．由此可得出 ω_E 与 r 的关系，于是第 1 问和第 3 问可解．它表明 r 的变化与 ω_E 的变化是有联系的．

地、月系统的总机械能 E 包括地球自转动能（与 ω_E 有关），月球绕地球转动的动能（与 ω_m 有关），以及地、月之间的引力势能（与 r 有关）．因此，$\frac{dE}{dt}$ 可用 $\dot{r}$ 表示，此即第 2 问．

月球引力引起的潮汐摩擦将损耗的机械能转变为热能，使地、月系统的总机械能 E 减少，导致地球自转角速度 ω_E 变小，地、月间距 r 加大．

Tips:
Hidden conditions that are most easily overlooked when doing planetary motion problems:

Angular momentum conservation and energy conservation

【解】 1. 地、月系统不受外力矩，角动量守恒，相对于地球的自转轴，有

$$I_E\omega_E + M_m r^2 \omega_m = \text{常量}$$

式中

$$I_E = \frac{2}{5} M_E R_E^2$$

为地球的转动惯量．对时间 t 求导，得

$$I_E\dot{\omega}_E + M_m \frac{d}{dt}(r^2\omega_m) = 0 \tag{1}$$

月球绕地球作圆周运动，由牛顿第二定律，得

$$G\frac{M_E M_m}{r^2} = M_m r \omega_m^2 \tag{2}$$

即

$$\omega_m = \left(\frac{GM_E}{r^3}\right)^{\frac{1}{2}}$$

代入(1)式，得

$$I_E\dot{\omega}_E + M_m\sqrt{GM_E}\frac{d}{dt}(r^{\frac{1}{2}}) = 0 \tag{3}$$

即

$$\frac{1}{2}r^{-\frac{1}{2}}\dot{r} = -\frac{I_E\dot{\omega}_E}{M_m\sqrt{GM_E}}$$

故

$$\dot{r} = -\frac{2\sqrt{r}\,I_E}{M_m\sqrt{GM_E}}\dot{\omega}_E \tag{4}$$

它表明，地球自转角速度 ω_E 的减小，将导致地、月间距 r 的增大．

Continue:

2. 地、月系统的机械能 E 包括地球自转动能，月球绕地球转动的动能以及地、月间的引力势能，为

$$E=\frac{1}{2}I_E\omega_E^2+\frac{1}{2}M_m r^2\omega_m^2-G\frac{M_E M_m}{r}$$

把(2)式代入，得

$$E=\frac{1}{2}I_E\omega_E^2-G\frac{M_E M_m}{2r}$$

机械能 E 随时间 t 的变化率为

$$\frac{dE}{dt}=I_E\omega_E\dot{\omega}_E+\frac{GM_E M_m}{2r^2}\dot{r}$$

把(4)式中的 $\dot{\omega}_E$ 代入，得

$$\frac{dE}{dt}=-\frac{M_m}{2}\left(\omega_E\sqrt{\frac{GM_E}{r}}-\frac{GM_E}{r^2}\right)\dot{r}$$

把基本常量值代入，可知上式圆括号内的量是正值．因此，当 $\dot{r}>0$ 时，$\frac{dE}{dt}<0$. 因潮汐摩擦，ω_E 将变小，由(4)式，r 要变大，故 E 将变小．

3. 由(3)式，

$$d\omega_E=-\frac{M_m}{I_E}\sqrt{GM_E}\,d(\sqrt{r})$$

积分，得

$$\omega_E(t)-\omega_E(0)=-\frac{M_m\sqrt{GM_E}}{I_E}\left[\sqrt{r(t)}-\sqrt{r(0)}\right]$$

或

$$\frac{\omega_E(t)}{\omega_E(0)}=1-\frac{M_m\sqrt{GM_E\,r(0)}}{I_E\,\omega_E(0)}\left[\sqrt{\frac{r(t)}{r(0)}}-1\right]$$

式中 $t=0$ 为现在时刻，$t=t$ 是将来某时刻．当

$$\frac{r(t)}{r(0)}=1.02$$

时，利用有关基本常量，得

$$\frac{\omega_E(t)}{\omega_0(t)}=0.96$$

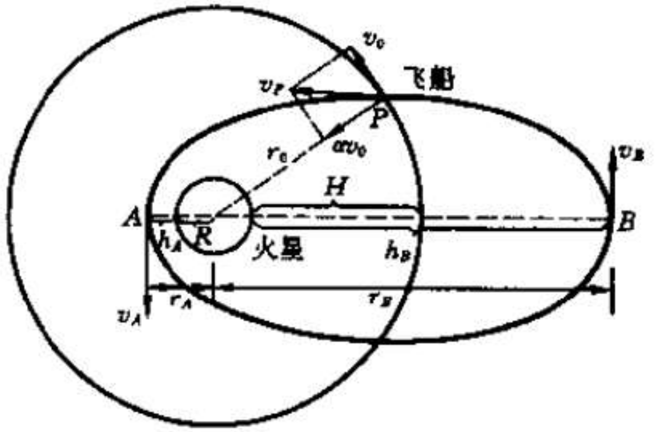
因此，当因潮汐摩擦使地、月间距比现在增加 2% 时，一个地球日（地球自转一圈的时间）将延长为

$$\frac{24}{0.96}=25\ \text{h}$$

Orbit Problems (most important)

As shown in the figure, a spaceship is orbiting Mars in a circular orbit with a speed of $v0$. It is known that the radius of Mars is R, and the height of the spaceship's circular orbit above the surface of Mars is H. Now, the spaceship fires a jet radially outward from the circular orbit in a very short period of time, so that the spaceship gains a radial velocity toward Mars of $av0$, where a is a constant much smaller than 1. Because the amount of gas ejected is very small, the mass of the spaceship can be considered unchanged after the jet. After the jet, the spaceship orbits Mars in a new elliptical orbit.

Find: 1. The height hA of the perigee of the spaceship's elliptical orbit from the surface of Mars and the height hB of the apogee of the elliptical orbit from the surface of Mars. 2. The orbital period of the spaceship in the elliptical orbit.

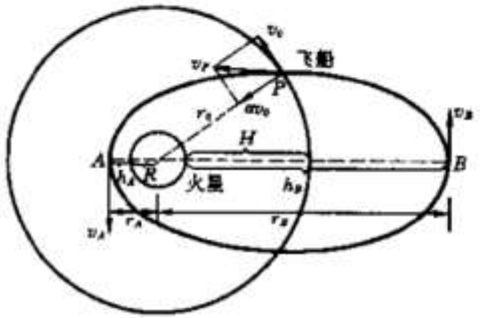


力图 4－11－1

Solution :

【题 11】 如图，宇宙飞船绕火星沿圆轨道运行，运动速度为 v_0. 已知火星半径为 R，飞船圆轨道离火星表面的高度为 H. 今飞船在极短时间内，沿圆轨道径向向外侧点火喷气，使飞船获得指向火星的径向速度 αv_0，α 是远小于 1 的常数．因喷气量很小，喷气后飞船的质量可视为不变．喷气后，飞船绕火星沿新的椭圆轨道运行．

试求:1. 飞船椭圆轨道近火星点距火星表面的高度 h_A 以及远火星点距火星表面的高度 h_B.2. 飞船绕椭圆轨道的运行周期．



力图 4－11－1

【分析】 飞船绕火星作圆运动时，圆轨道半径为 $r_0 = R + H$，面积速度为 $\frac{1}{2} r_0 v_0$. 飞船沿圆轨道径向向外侧喷气后，飞船将沿新的以火星为焦点的椭圆轨道运行，设近火星点为 A，远火星点为 B，A 和 B 与火星中心的距离分别为 r_A 和 r_B（$r_A = h_A + R$，$r_B = h_B + R$）. 飞船沿椭圆轨道运行时，面积速度恒定，机械能守恒．

应特别注意，飞船沿圆轨道径向向外侧喷气后，使之获得指向火星的径向速度 αv_0，因喷气方向与飞船原先绕火星作圆轨道运动的速度 $\boldsymbol{v}_0$ 垂直，故喷气并不改变 $\boldsymbol{v}_0$ 的大小和方向，喷气后飞船的速度 $\boldsymbol{v}_p$ 是 $\boldsymbol{v}_0$ 与 $\alpha \boldsymbol{v}_0$（其方向与 $\boldsymbol{v}_0$ 垂直）的矢量和．换言之，$v_0 = v_p \sin\theta$，故短时间喷气后，飞船绕椭圆轨道的面积速度为 $\frac{1}{2} r_0 v_p \sin\theta = \frac{1}{2} r_0 v_0$，就等于飞船喷气前绕圆轨道运行的面积速度，亦即喷气前后飞船的轨道虽然改变了，但面积速度不变．由飞船沿椭圆轨道运行时，面积速度恒定及机械能守恒两条规律可以解出 r_A 和 r_B，再由开普勒第三定律可求出飞船绕椭圆轨道的运行周期．

Solution :

【解】 设火星和飞船的质量分别为 M 和 m，飞船沿椭圆轨道运行时，飞船与火星中心的距离统一用 r 表示，飞船的速度统一用 v 表示，再加下标注明飞船所在位置．

根据分析，飞船喷气前绕圆轨道运行的面积速度 $\frac{1}{2}r_0v_0$ 等于喷气后飞船绕椭圆轨道运行在 P 点的面积速度 $\frac{1}{2}r_0v_p\sin\theta$（$P$ 点是圆和椭圆的交点）．由开普勒第二定律，后者又应等于飞船在近火星点 A 和远火星点 B 的面积速度 $\frac{1}{2}r_Av_A$ 和 $\frac{1}{2}r_Bv_B$．故有

$$\frac{1}{2}r_0v_0=\frac{1}{2}r_0v_p\sin\theta=\frac{1}{2}r_Av_A=\frac{1}{2}r_Bv_B$$

即

$$r_0v_0=r_Av_A=r_Bv_B \tag{1}$$

由机械能守恒定律，有

$$\frac{1}{2}mv_p^2-\frac{GMm}{r_p}=\frac{1}{2}mv_A^2-\frac{GMm}{r_A}=\frac{1}{2}mv_B^2-\frac{GMm}{r_B}$$

因

$$v_p^2=v_0^2+(\alpha v_0)^2,\quad r_p=r_0$$

由以上三式，有

$$\frac{1}{2}m[v_0^2+(\alpha v_0)^2]-\frac{GMm}{r_0}=\frac{1}{2}mv_A^2-\frac{GMm}{r_A}=\frac{1}{2}mv_B^2-\frac{GMm}{r_B} \tag{2}$$

飞船沿原圆轨道运行时，有

$$\frac{GMm}{r_0^2}=\frac{mv_0^2}{r_0}$$

$$GM=r_0v_0^2 \tag{3}$$

联立(1)、(2)、(3)式，可得出关于 r（指 r_A 或 r_B）的二次方程式为

$$(1-\alpha^2)r^2-2r_0r+r_0^2=0$$

上式有两个解，大者就是 r_B，小者就是 r_A，为

$$\begin{cases} r_A=\dfrac{r_0}{1+\alpha}=\dfrac{R+H}{1+\alpha} \\ r_B=\dfrac{r_0}{1-\alpha}=\dfrac{R+H}{1-\alpha} \end{cases}$$

r_A 和 r_B 分别是飞船近火星点和远火星点与火星中心的距离．故近火星点和远火星点距火星表面的高度分别为

$$\begin{cases} h_A=r_A-R=\dfrac{H-\alpha R}{1+\alpha} \\ h_B=r_B-R=\dfrac{H+\alpha R}{1-\alpha} \end{cases}$$

设飞船椭圆轨道的半长轴为 a，则

$$r_A+r_B=2a$$

即

$$a=\frac{1}{2}(r_A+r_B)=\frac{r_0}{1-\alpha}$$

飞船喷气前绕圆轨道运行的周期为

$$T_0=\frac{2\pi r_0}{v_0}$$

设飞船喷气后绕椭圆轨道运行的周期为 T，由开普勒第三定律，有

$$\frac{T}{T_0}=\left(\frac{a}{r_0}\right)^{3/2}$$

故

$$T=T_0\left(\frac{a}{r_0}\right)^{3/2}=\frac{2\pi r_0}{v_0}\left(\frac{1}{1-\alpha^2}\right)^{3/2}$$

Thank you !

From Mass Effect Legendary Edition

Grade is not everything !

Hope you can enjoy Physics itself !!!

From Watch Dog2

最后我们都会共坐古舟，行至人间
那些沿途的春秋冷暖，再见已难
临水忘川，总觉得月落比日出更好看
从容些
生活会更好

——司徒登